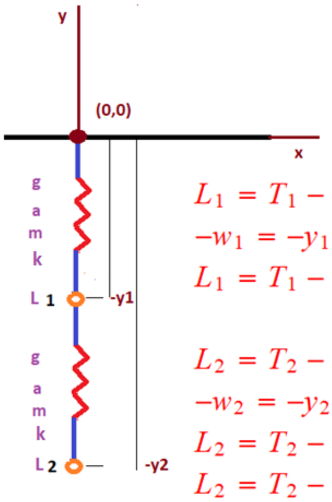
MECÁNICA TEÓRICA EJERCICIO DE MUELLES CON GRAVEDAD:

a) Ejes de referencia y primer cambio de variables (w1,w2):



$$L_1 = T_1 - V_1 = m/2 * (-y_1)^2 - k/2 * (-y_1 - (-a))^2 - mg(-y_1)$$
$$-w_1 = -y_1 - (-a); w_1 = y_1 - a; y_1 = w_1 + a;\ \dot{y}_1 = \dot{w}_1$$
$$L_1 = T_1 - V_1 = m/2 * \dot{w}_1{}^2 - k/2 * w_1{}^2 + mg(w_1 + a)$$

$$L_2 = T_2 - V_2 = m/2 * (-y_2)^2 - k/2 * (-y_2 - (-y_1) - (-a))^2 - mg(-y_2)$$
$$-w_2 = -y_2 - (-2a) = -y_2 + 2a; w_2 = y_2 - 2a; y_2 = w_2 + 2a;\ \dot{y}_2 = \dot{w}_2$$
$$L_2 = T_2 - V_2 = m/2 * \dot{w}_2{}^2 - k/2 * (-w_2 - 2a + w_1 + a + a)^2 + mg(w_2 + 2a)$$
$$L_2 = T_2 - V_2 = m/2 * \dot{w}_2{}^2 - k/2 * (-w_2 + w_1)^2 + mg(w_2 + 2a)$$

b) Lagrangiano de los dos móviles y segundo cambio de variables (z1,z2):

$$L = L_1 + L_2$$
$$L = m/2 * [\dot{w}_1{}^2 + \dot{w}_2{}^2] - k/2 * [w_1{}^2 + (-w_2 + w_1)^2] + mg * [w_1 + a + w_2 + 2a]$$
$$L = m/2 * [\dot{w}_1{}^2 + \dot{w}_2{}^2] - k/2 * [w_1{}^2 + (w_2 - w_1)^2] + mg * [w_1 + w_2 + 3a]$$

Matriz A, D(autovalores), M (ortonormal), MT y M⁻¹:

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \text{ eigenvalues: } \tfrac{1}{2}\sqrt{5} + \tfrac{3}{2}, \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}$$

$$\left\{\begin{bmatrix} -\tfrac{1}{2}\sqrt{5} + \tfrac{1}{2} \\ 1 \end{bmatrix}\right\} \leftrightarrow \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}, \left\{\begin{bmatrix} \tfrac{1}{2}\sqrt{5} + \tfrac{1}{2} \\ 1 \end{bmatrix}\right\} \leftrightarrow \tfrac{1}{2}\sqrt{5} + \tfrac{3}{2}, \text{eigenvectors:}$$

$$\begin{bmatrix} -\tfrac{1}{2}\sqrt{5} - \tfrac{1}{2} & \tfrac{1}{2}\sqrt{5} - \tfrac{1}{2} \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} \sqrt{(\sqrt{5} + 5)/10} & \sqrt{(5 - \sqrt{5})/10} \\ -\sqrt{(5 - \sqrt{5})/10} & \sqrt{(\sqrt{5} + 5)/10} \end{bmatrix}$$

ORTONORMAL

Comprobación MT = M⁻¹:

$$\begin{bmatrix} \sqrt{(\sqrt{5} + 5)/10} & \sqrt{(5 - \sqrt{5})/10} \\ -\sqrt{(5 - \sqrt{5})/10} & \sqrt{(\sqrt{5} + 5)/10} \end{bmatrix}, \text{ inverse: } \begin{bmatrix} \sqrt{\tfrac{1}{10}\sqrt{5} + \tfrac{1}{2}} & -\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} \\ \sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} & \sqrt{\tfrac{1}{10}\sqrt{5} + \tfrac{1}{2}} \end{bmatrix}$$

Expresión producto escalar (w1,w2)*A* (w1,w2):

$$[{w_1}^2+(w_2-w_1)^2] = {w_1}^2+{w_2}^2+{w_1}^2-2w_1w_2 = 2{w_1}^2+{w_2}^2-2w_1w_2$$

$$(w_1,w_2)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 2{w_1}^2+{w_2}^2-2w_1w_2$$

Cambio de variables (w1,w2) = M*(z1,z2):

$$\begin{pmatrix} \sqrt{(\sqrt{5}+5)/10} & \sqrt{(5-\sqrt{5})/10} \\ -\sqrt{(5-\sqrt{5})/10} & \sqrt{(5+\sqrt{5})/10} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1\sqrt{\frac{1}{10}\sqrt{5}+\frac{1}{2}}+z_2\sqrt{\frac{1}{2}-\frac{1}{10}\sqrt{5}} \\ -z_1\sqrt{\frac{1}{2}-\frac{1}{10}\sqrt{5}}+z_2\sqrt{\frac{1}{10}\sqrt{5}+\frac{1}{2}} \end{pmatrix}$$

$$w_1 = z_1\sqrt{\frac{1}{10}\sqrt{5}+\frac{1}{2}}+z_2\sqrt{\frac{1}{2}-\frac{1}{10}\sqrt{5}}$$

$$w_2 = -z_1\sqrt{\frac{1}{2}-\frac{1}{10}\sqrt{5}}+z_2\sqrt{\frac{1}{10}\sqrt{5}+\frac{1}{2}}$$

Productos escalares en las dos bases son iguales (w1,w2)*A*(w1,w2) = (z1,z2)*D*(z1,z2) **y** $\dot{w}_1=\dot{z}_1$; $\dot{w}_2=\dot{z}_2$:

$$L = m/2*[{\dot z_1}^2+{\dot z_2}^2]-k/2*\left[{z_1}^2\left(\tfrac{1}{2}\sqrt{5}+\tfrac{3}{2}\right)+{z_2}^2\left(\tfrac{3}{2}-\tfrac{1}{2}\sqrt{5}\right)\right]$$

$$+mg*\left[z_1\left(\sqrt{\tfrac{1}{10}\sqrt{5}+\tfrac{1}{2}}-\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}}\right)+z_2\left(\sqrt{\tfrac{1}{10}\sqrt{5}+\tfrac{1}{2}}+\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}}\right)+3a\right]$$

Simplificación de términos:

$$\sqrt{\tfrac{1}{10}\sqrt{5}+\tfrac{1}{2}}-\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}} = \tfrac{1}{2}\left(\sqrt{5}-1\right)\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}}$$

$$\sqrt{\tfrac{1}{10}\sqrt{5}+\tfrac{1}{2}}+\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}} = \tfrac{1}{2}\left(\sqrt{5}+3\right)\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}}$$

Lagrangiano con las nuevas variables (z1,z2):

$$L = m/2*[{\dot z_1}^2+{\dot z_2}^2]-k/2*\left[{z_1}^2\left(\tfrac{1}{2}\sqrt{5}+\tfrac{3}{2}\right)+{z_2}^2\left(\tfrac{3}{2}-\tfrac{1}{2}\sqrt{5}\right)\right]$$

$$+mg*\left[z_1\left(\sqrt{\tfrac{1}{10}\sqrt{5}+\tfrac{1}{2}}-\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}}\right)+z_2\left(\sqrt{\tfrac{1}{10}\sqrt{5}+\tfrac{1}{2}}+\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}}\right)+3a\right]$$

$$L = m/2*[{\dot z_1}^2+{\dot z_2}^2]-k/2*\left[{z_1}^2\left(\tfrac{1}{2}\sqrt{5}+\tfrac{3}{2}\right)+{z_2}^2\left(\tfrac{3}{2}-\tfrac{1}{2}\sqrt{5}\right)\right]$$

$$+mg*\left[z_1\ \tfrac{1}{2}\left(\sqrt{5}-1\right)\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}}+z_2\ \tfrac{1}{2}\left(\sqrt{5}+3\right)\sqrt{\tfrac{1}{2}-\tfrac{1}{10}\sqrt{5}}+3a\right]$$

c) **Ecuación Euler – Lagrange con el segundo cambio de variables** (q1=z1,q2=z2):

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} = 0$$

$$m * \ddot{z}_1 + k * z_1 \left(\tfrac{1}{2}\sqrt{5} + \tfrac{3}{2}\right) - mg * \tfrac{1}{2}\left(\sqrt{5} - 1\right)\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} = 0$$

$$m * \ddot{z}_2 + k * z_2 \left(\tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}\right) - mg * \tfrac{1}{2}\left(\sqrt{5} + 3\right)\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} = 0$$

$$\ddot{z}_1 + k/m * z_1 \left(\tfrac{1}{2}\sqrt{5} + \tfrac{3}{2}\right) - g * \tfrac{1}{2}\left(\sqrt{5} - 1\right)\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} = 0$$

$$\ddot{z}_2 + k/m * z_2 \left(\tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}\right) - g * \tfrac{1}{2}\left(\sqrt{5} + 3\right)\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} = 0$$

$$\ddot{z}_1 + k/2\,m\left(\sqrt{5} + 3\right) * z_1 - g * \tfrac{1}{2}\left(\sqrt{5} - 1\right)\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} = 0$$

$$\ddot{z}_2 + k/2\,m\left(3 - \sqrt{5}\right) * z_2 - g * \tfrac{1}{2}\left(\sqrt{5} + 3\right)\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} = 0$$

d) **Resolución de la Ecuación diferencial** :

$y'' + cy + d = 0$, Exact solution is:

$$\left\{\tfrac{1}{c}C_2\left(c\cos\sqrt{c}\,t - ic\sin\sqrt{c}\,t\right) - \tfrac{1}{c}d + \tfrac{1}{c}C_1\left(c\cos\sqrt{c}\,t + ic\sin\sqrt{c}\,t\right)\right\}$$

$$y(t) = (C_1 + C_2)\cos\sqrt{c}\,t - (C_1 - C_2)i\sin\sqrt{c}\,t - \tfrac{1}{c}d$$

$$C_3 = C_1 + C_2 = \cos\beta$$

$$C_4 = (C_1 - C_2)i = \sin\beta$$

$$y(t) = \cos\beta\cos\sqrt{c}\,t - \sin\beta\sin\sqrt{c}\,t - \tfrac{1}{c}d = \cos(\sqrt{c}\,t + \beta) - \tfrac{1}{c}d$$

$$y(t) = \cos(\sqrt{c}\,t + \beta) - \tfrac{1}{c}d$$

Aplicación resolución con las nuevas variables (z1,z2):

$$\ddot{z}_1 + k/2\,m\underbrace{(\sqrt{5}+3)}_{c_1} * z_1 + \underbrace{\left(-g * \tfrac{1}{2}(\sqrt{5} - 1)\right)}_{d_1}\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} = 0$$

$$z_1(t) = \cos(\sqrt{c_1}\,t + \beta_1) - \tfrac{1}{c_1}d_1$$

$$\ddot{z}_2 + k/2\,m\underbrace{(3-\sqrt{5})}_{c_2} * z_2 + \underbrace{\left(-g * \tfrac{1}{2}(\sqrt{5} + 3)\right)}_{d_2}\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} = 0$$

$$z_2(t) = \cos(\sqrt{c_2}\,t + \beta_2) - \tfrac{1}{c_2}d_2$$

e) Solución con las segundas variables (w1,w2):

$$w_1 = z_1\sqrt{\tfrac{1}{10}\sqrt{5} + \tfrac{1}{2}} + z_2\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}}$$

$$w_2 = -z_1\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}} + z_2\sqrt{\tfrac{1}{10}\sqrt{5} + \tfrac{1}{2}}$$

f) Solución con las primeras variables (y1,y2):

$$y_1(t) = (\cos(\sqrt{c_1}\,t + \beta_1) - \tfrac{1}{c_1}d_1)\sqrt{\tfrac{1}{10}\sqrt{5} + \tfrac{1}{2}}) + (\cos(\sqrt{c_2}\,t + \beta_2) - \tfrac{1}{c_2}d_2)\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}}) + a$$

$$y_2(t) = -(\cos(\sqrt{c_1}\,t + \beta_1) - \tfrac{1}{c_1}d_1)\sqrt{\tfrac{1}{2} - \tfrac{1}{10}\sqrt{5}}) + (\cos(\sqrt{c_2}\,t + \beta_2) - \tfrac{1}{c_2}d_2)\sqrt{\tfrac{1}{10}\sqrt{5} + \tfrac{1}{2}}) + 2a$$

$$y_1(t) = (\cos(\sqrt{c_1}\,t + \beta_1) - \tfrac{1}{c_1}d_1)0.85065) + (\cos(\sqrt{c_2}\,t + \beta_2) - \tfrac{1}{c_2}d_2)0.52573) + a$$

$$y_2(t) = -(\cos(\sqrt{c_1}\,t + \beta_1) - \tfrac{1}{c_1}d_1)0.52573) + (\cos(\sqrt{c_2}\,t + \beta_2) - \tfrac{1}{c_2}d_2)0.85065) + 2a$$

Simplificación de términos:

$$y_1(t) = 0.85065\cos(1.618\sqrt{k/m}\,t + \beta_1) + 0.52573\cos(0.61804\sqrt{k/m}\,t + \beta_2) + \tfrac{2}{(k/m)}g + a$$

$$y_2(t) = -0.52573\cos(1.618\sqrt{k/m}\,t + \beta_1) + 0.85065\cos(0.61804\sqrt{k/m}\,t + \beta_2) + \tfrac{3}{(k/m)}g + 2a$$

g) Código Matlab R2014a:

```
        clc, clear, close all

A = 0 ; B = 0 ; C = 1 ; D = 1 ;
m = 0.15;
k = 10;
wo = sqrt(k/m);
a = 3;
g = 9.81;

xlim([-8 8])
ylim([-8 8])
axis square

for t=0:0.6:100

    y1 = 0;
    y2 = -(a +2*g/wo^2);
    y2 = y2+C*0.85065*cos(1.618*wo*t+A)+D*0.52573*cos(0.61804*wo*t+B);
    y3 = -(2*a +2*g/wo^2);
    y3 = y3-C*0.52573*cos(1.618*wo*t+A)+D*0.85065*cos(0.61804*wo*t+B);

    line([0 0 0],[y1 y2 y3],'Color','blue')

    viscircles([0 y1],0.1,'EdgeColor','red');
    viscircles([0 y2],0.1,'EdgeColor','red');
    viscircles([0 y3],0.1,'EdgeColor','red');

    pause(0.1)

    viscircles([0 y1],0.1,'EdgeColor','white');
    viscircles([0 y2],0.1,'EdgeColor','white');
    viscircles([0 y3],0.1,'EdgeColor','white');

    line([0 0 0],[y1 y2 y3],'Color','white')

end
```